Descriptive Complexity of Graph Classes via Labeling Schemes

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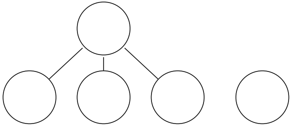
FernUniversität in Hagen

ACTO Seminar
University of Liverpool
May 2021

What is a Labeling Scheme?



interval graph G

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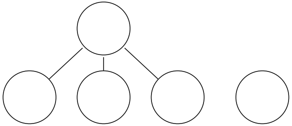

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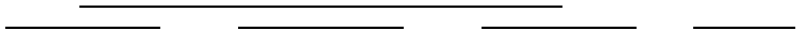

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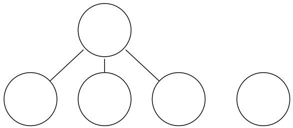

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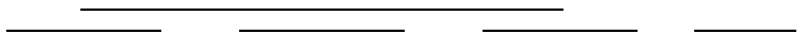

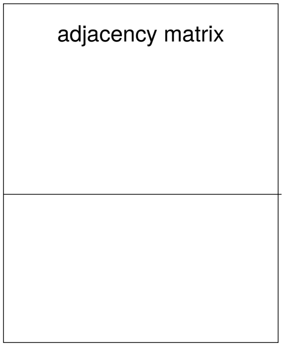

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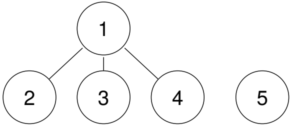

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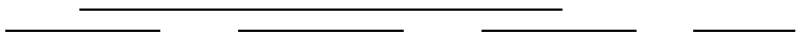

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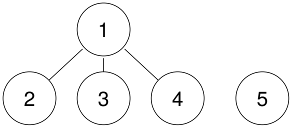

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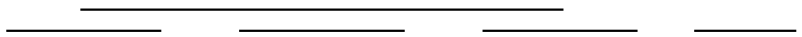

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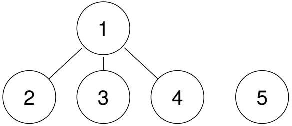

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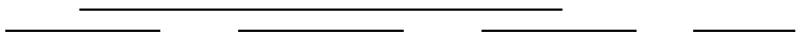

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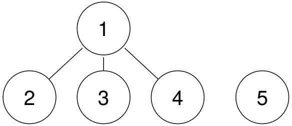

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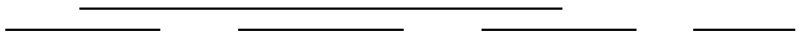

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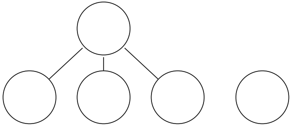

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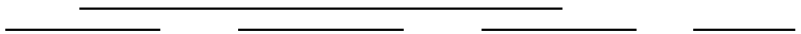

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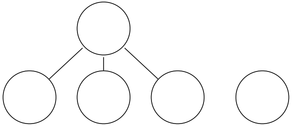

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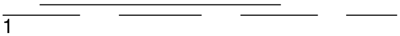

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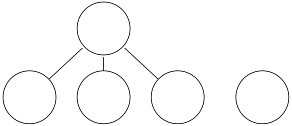

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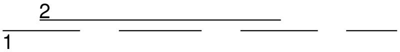

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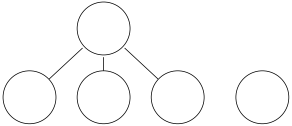

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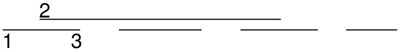

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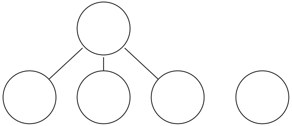

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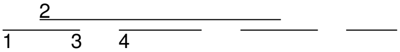

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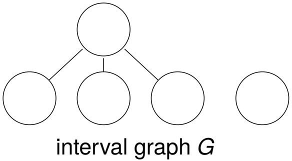

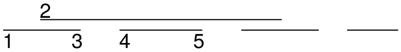

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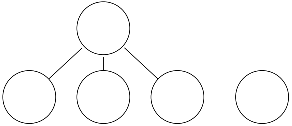

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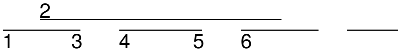

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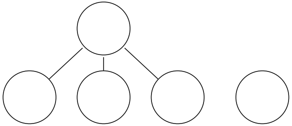

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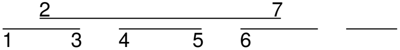

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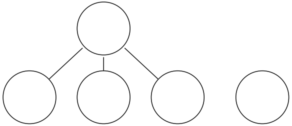

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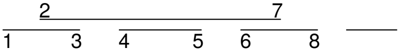

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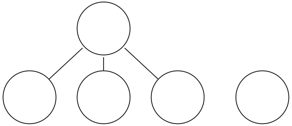

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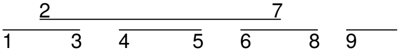

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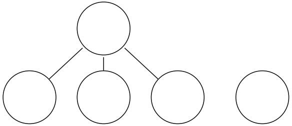

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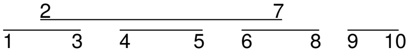

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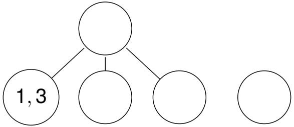

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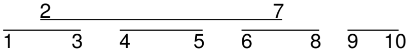

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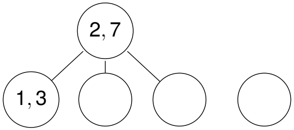

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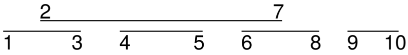

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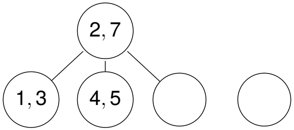

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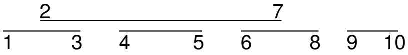

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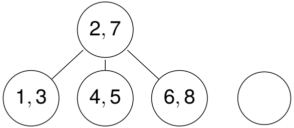

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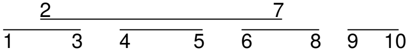

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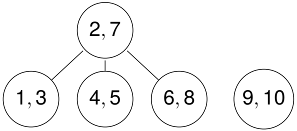

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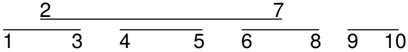

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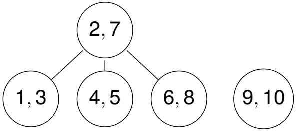

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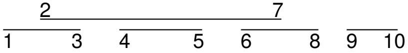

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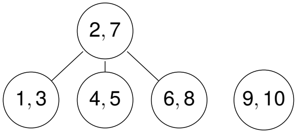

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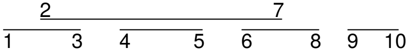

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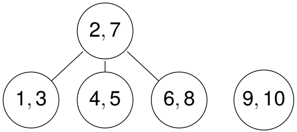

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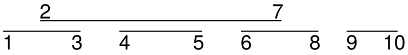

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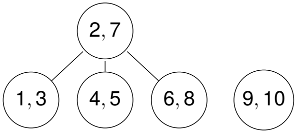

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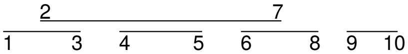

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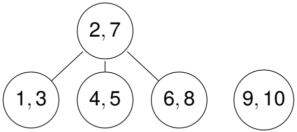

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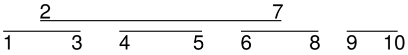

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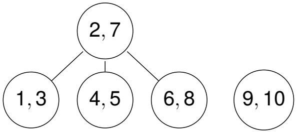

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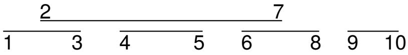

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labeling $\ell\colon V(G) \to \{0,1\}^{2(1+\log n)}$

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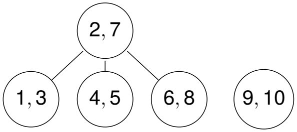

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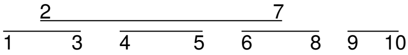

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requires n^2 bits

alternative representation

$\{(1,3),(2,7),(4,5),\dots\}$

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u

v

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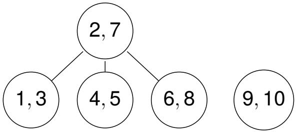

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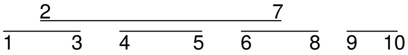

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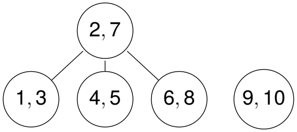

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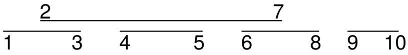

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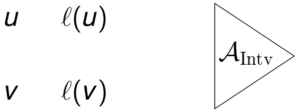

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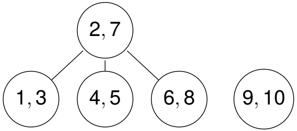

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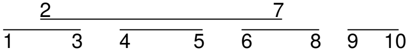

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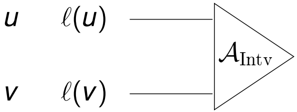

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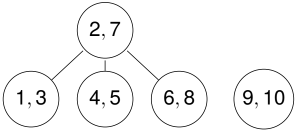

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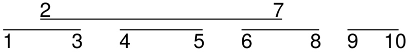

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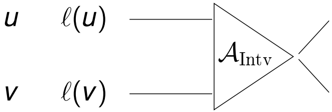

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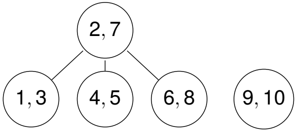

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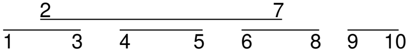

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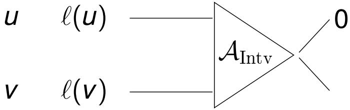

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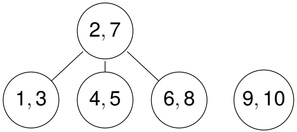

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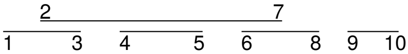

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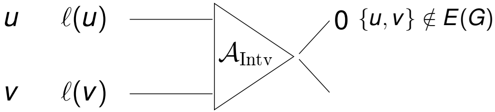

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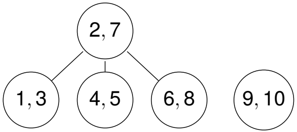

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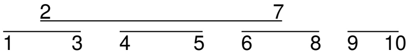

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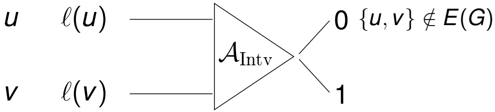

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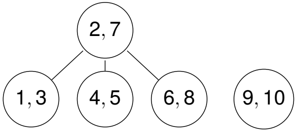

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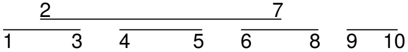

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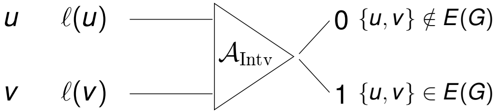

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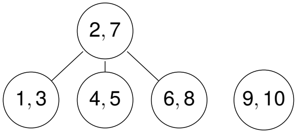

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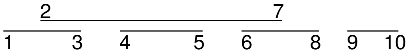

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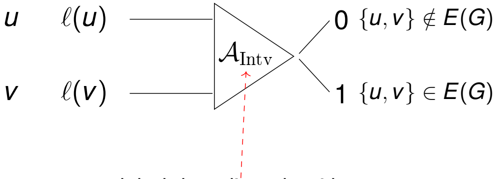

label decoding algorithm

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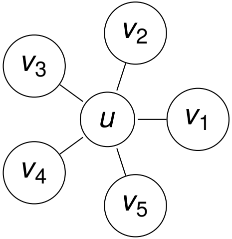

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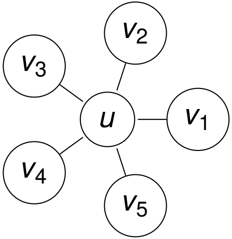

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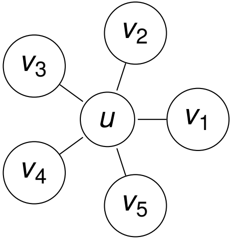

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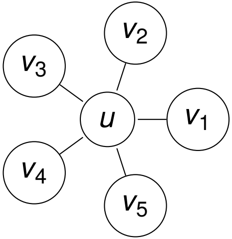

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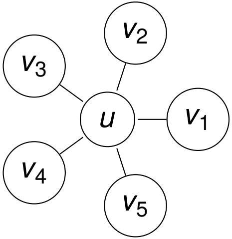

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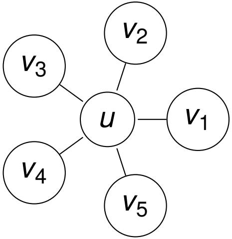

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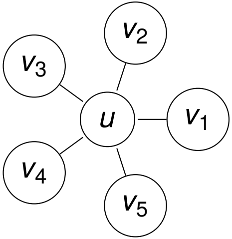

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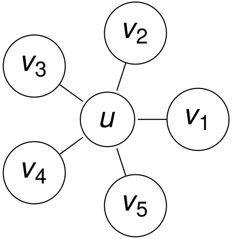

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- extensions: distance labeling schemes, variable label length, ...

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what graph classes have a labeling scheme
when label decoder F has low complexity?

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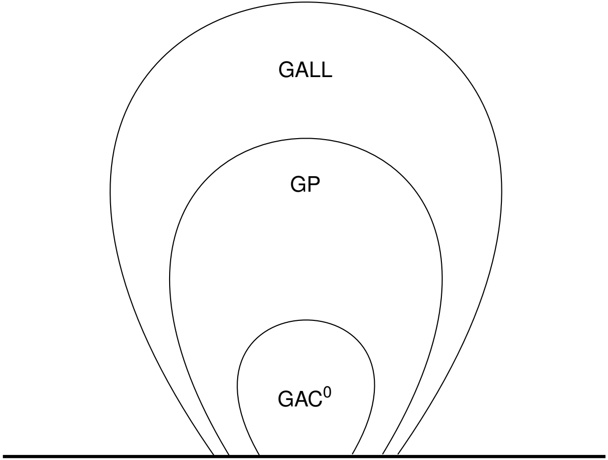

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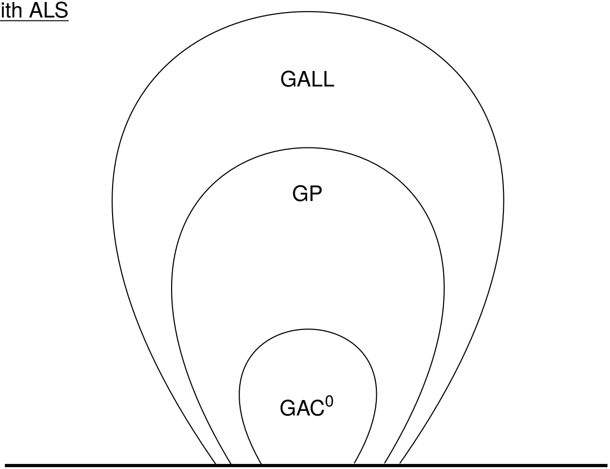
some classes with ALS



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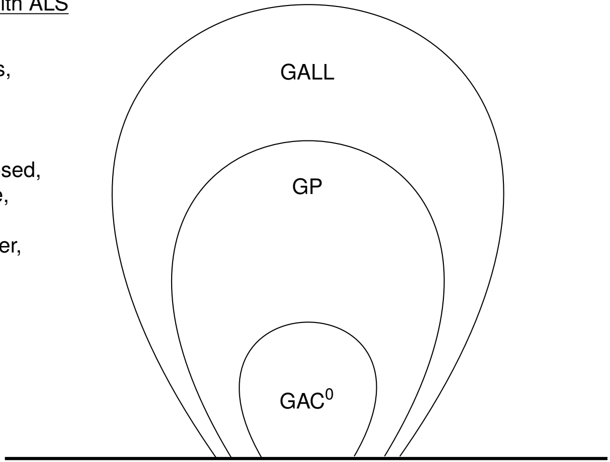

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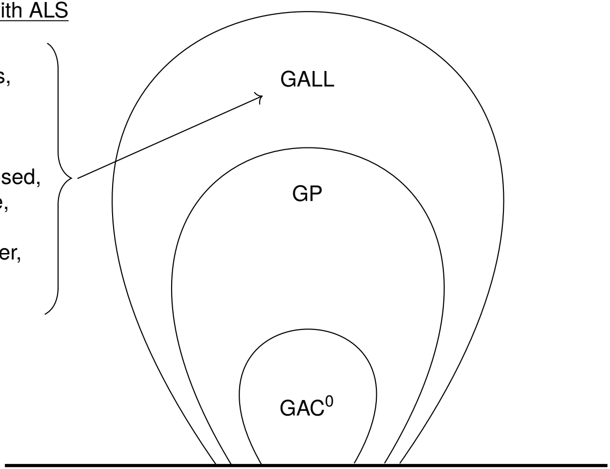

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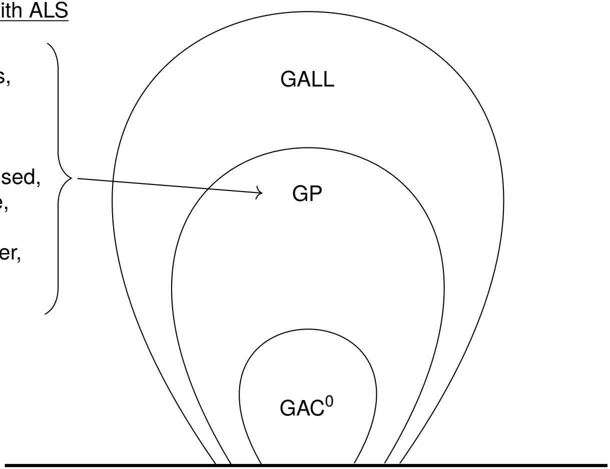

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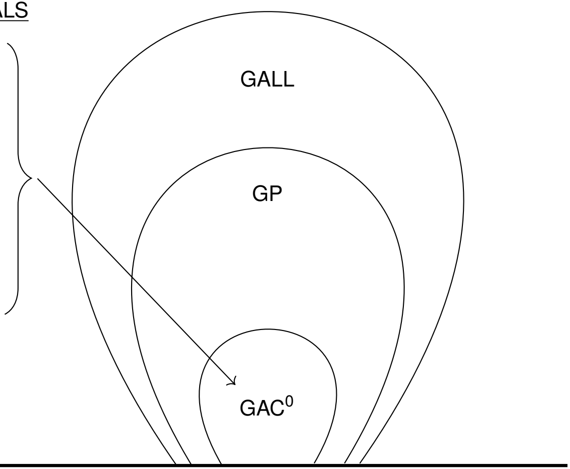

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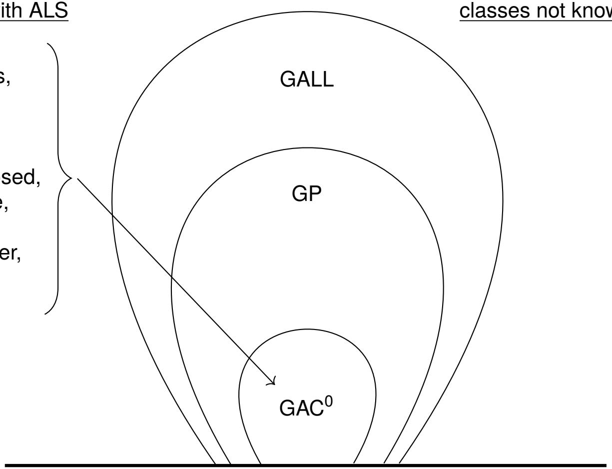

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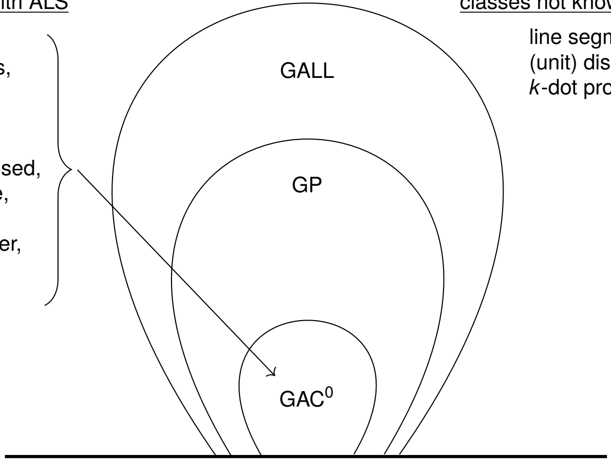

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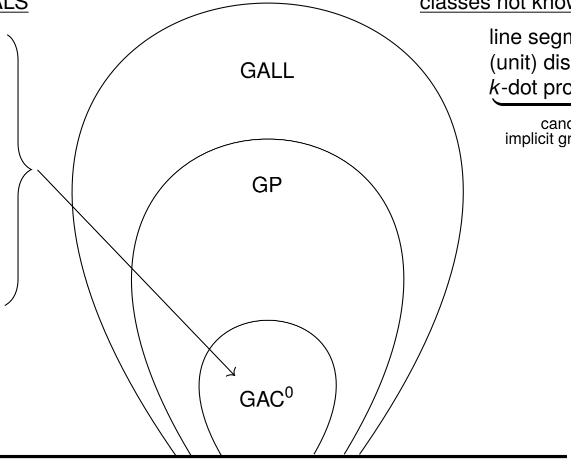

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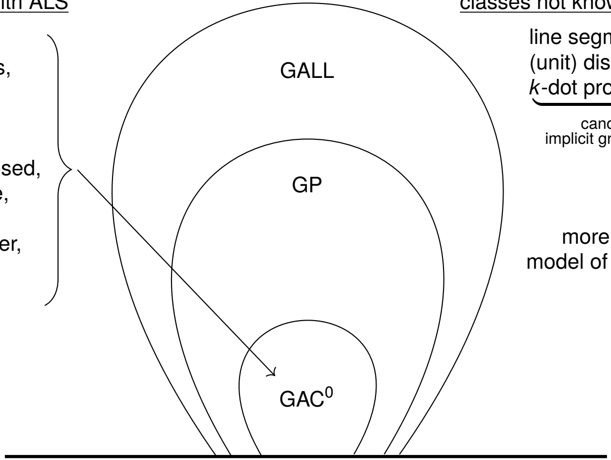

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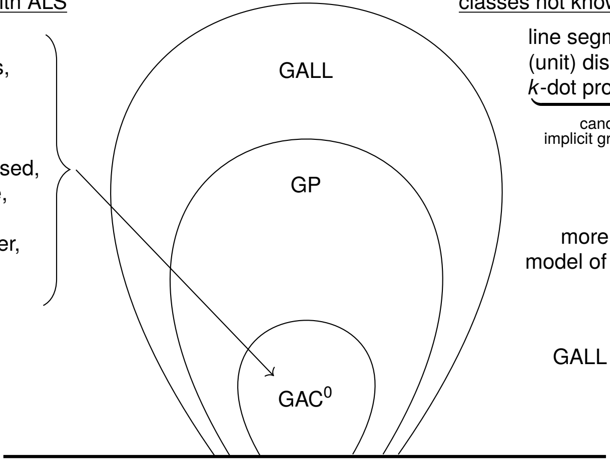

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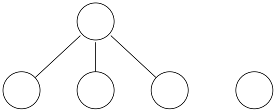

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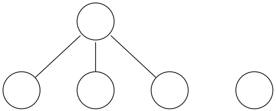

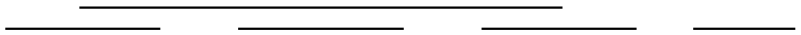

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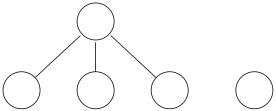

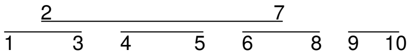

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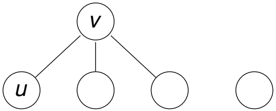

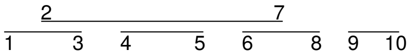

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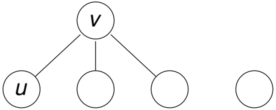

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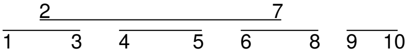

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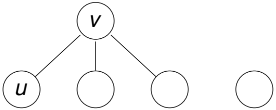

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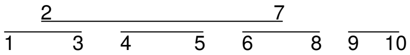

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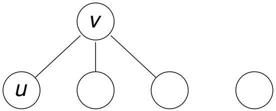

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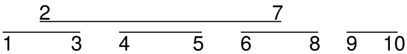

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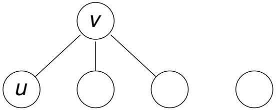

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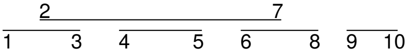

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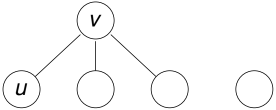

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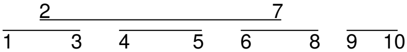

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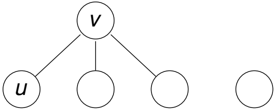

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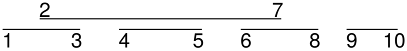
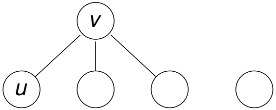


$$\varphi_{\mathrm{Intv}}(u_1, u_2, v_1, v_2) \triangleq \neg(u_2 < v_1 \vee v_2 < u_1)$$

- logical labeling scheme (φ, k, c)
  - φ: FO formula using $=, <, +, \times$

Logical Labeling Schemes



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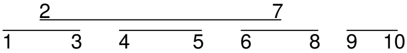

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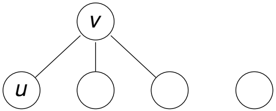

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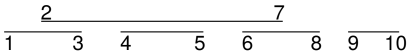

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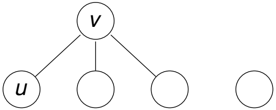

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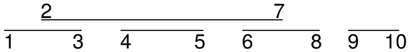

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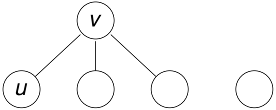

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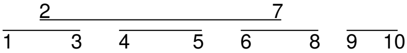

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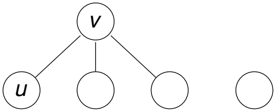
Logical Labeling Schemes



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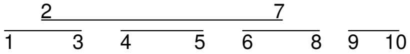

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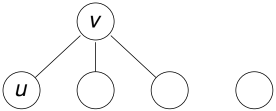

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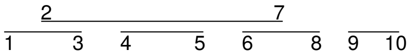

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- classes $\mathrm{GFO}(\sigma)$ and $\mathrm{GFO}_{\mathrm{qf}}(\sigma)$ for $\sigma \subseteq \{<, +, \times\}$
  - interval graphs in $\mathrm{GFO}_{\mathrm{qf}}(<)$

Logical Complexity Classes

Logical Complexity Classes

GFO$(<, +, \times)$

Logical Complexity Classes

$\text{GFO}(<, +, \times)$

$|$

$\text{GFO}_{\text{qf}}(<, +, \times)$

Logical Complexity Classes

$\mathsf{GFO}(<, +, \times)$

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Logical Complexity Classes

$\mathrm{GFO}(<, +, \times)$

$\mathrm{GFO}_{\mathrm{qf}}(<, +, \times)$

$\mathrm{GFO}_{\mathrm{qf}}(<, +) = \mathrm{GFO}_{\mathrm{qf}}(<, \times)$

Logical Complexity Classes

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Logical Complexity Classes

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Logical Complexity Classes

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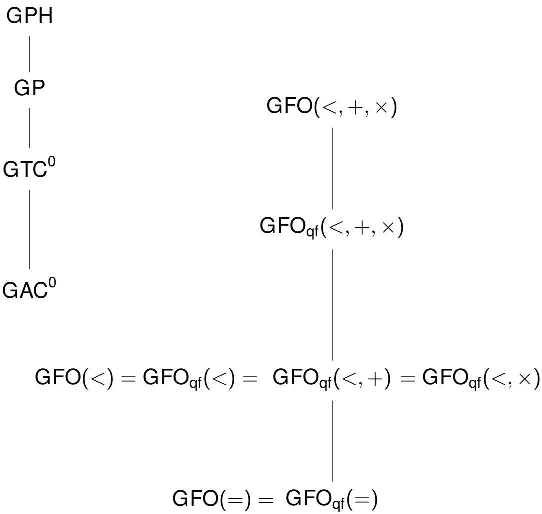
Logical Complexity Classes



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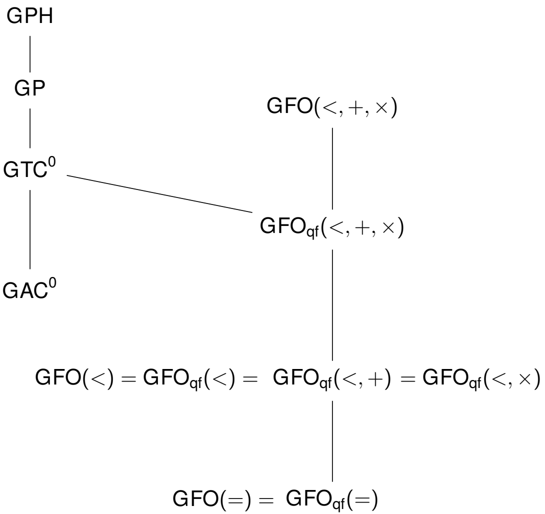

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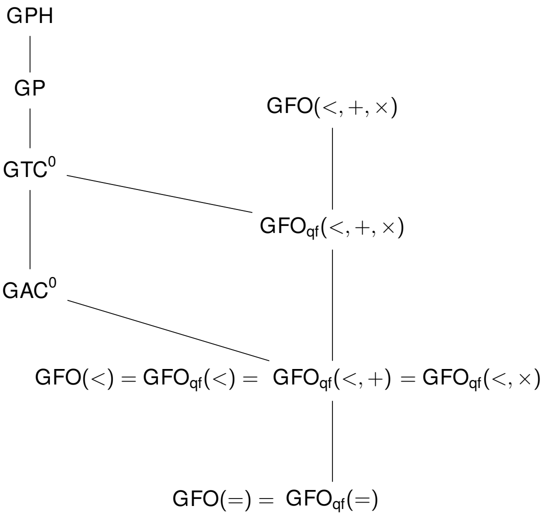

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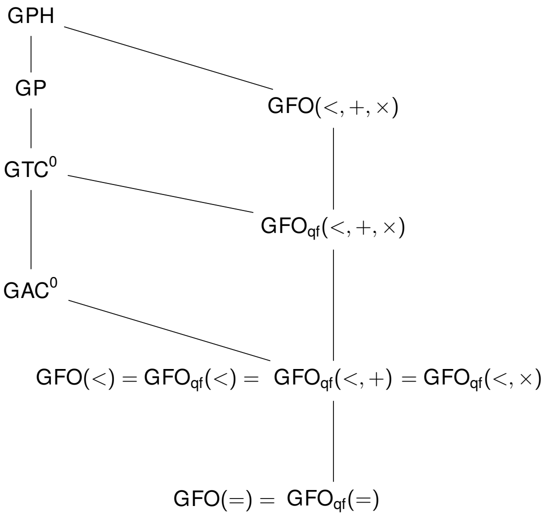

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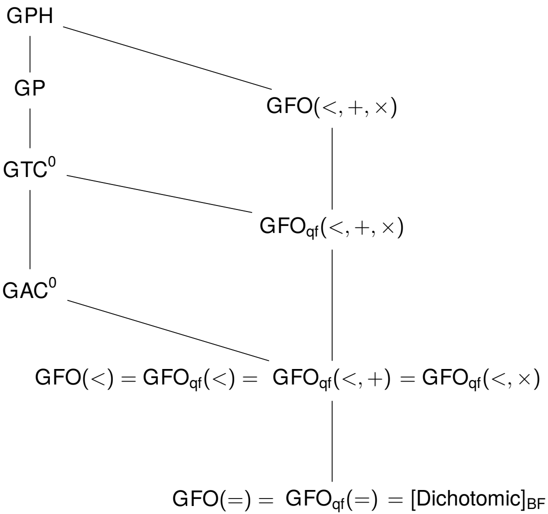

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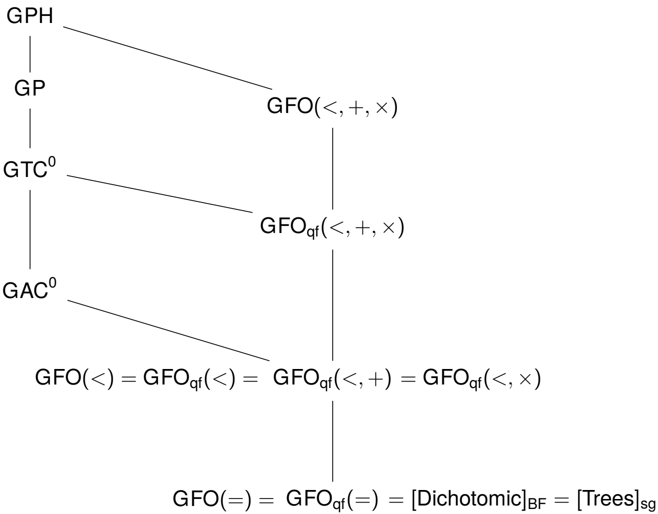

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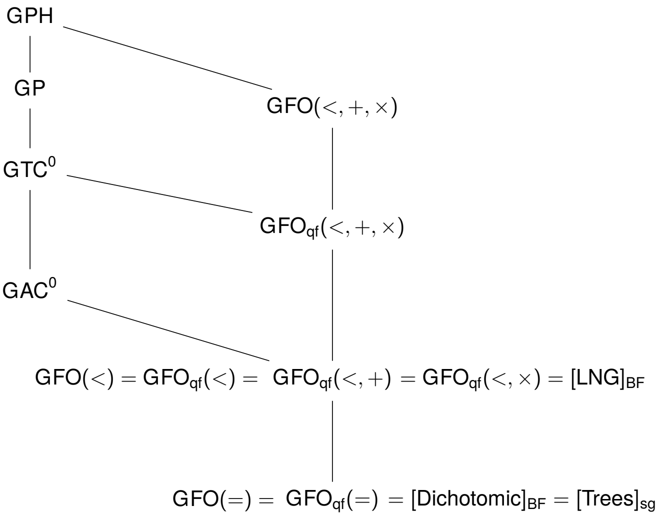

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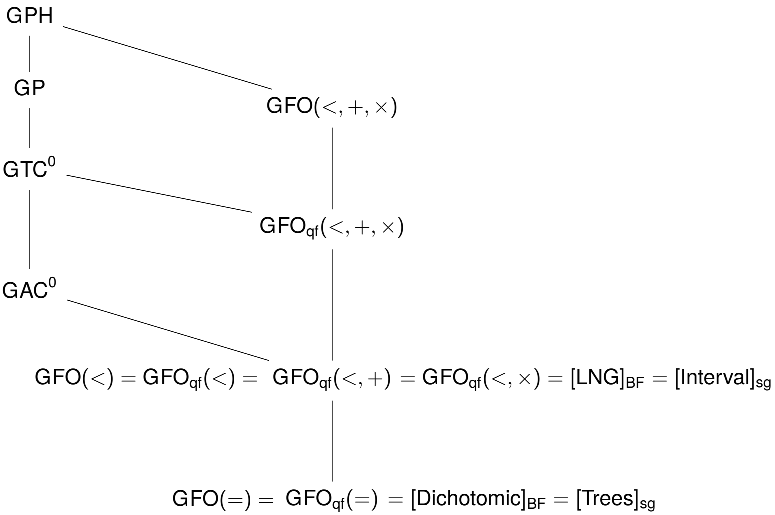

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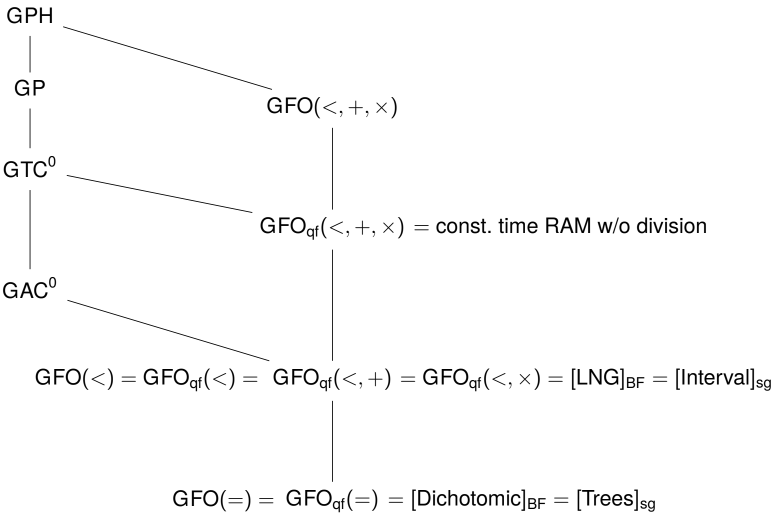

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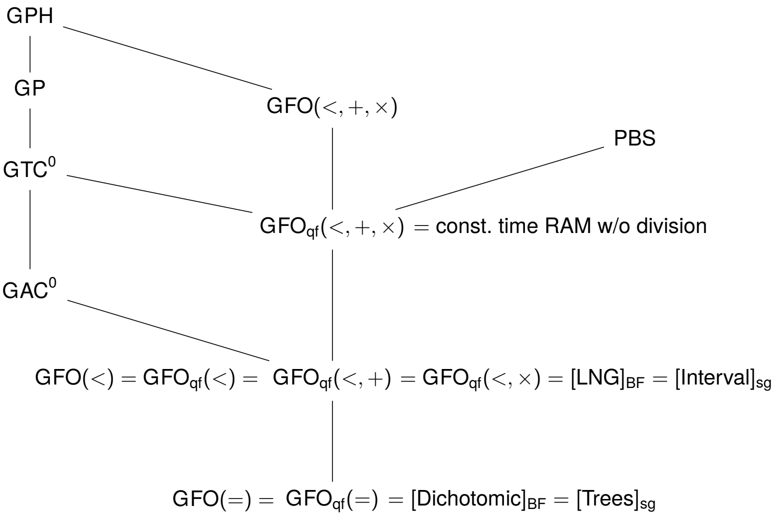

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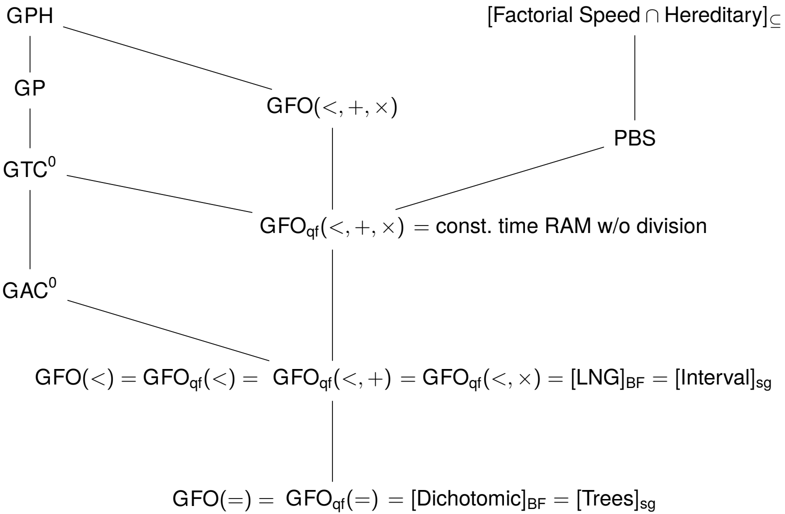

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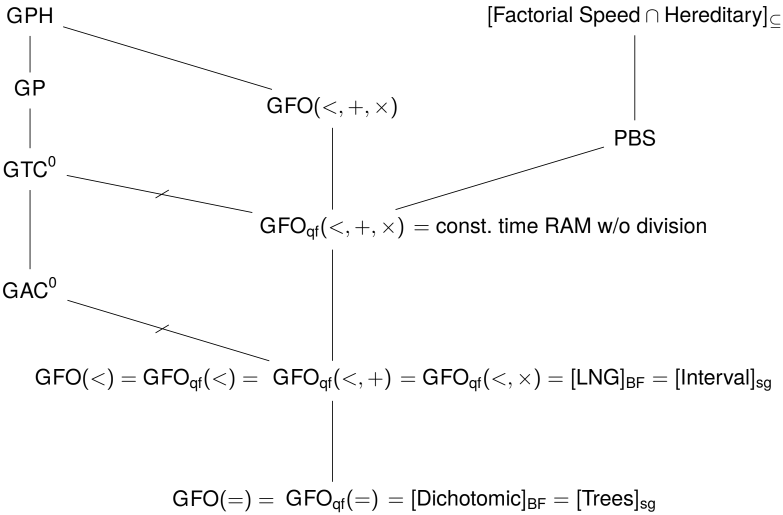

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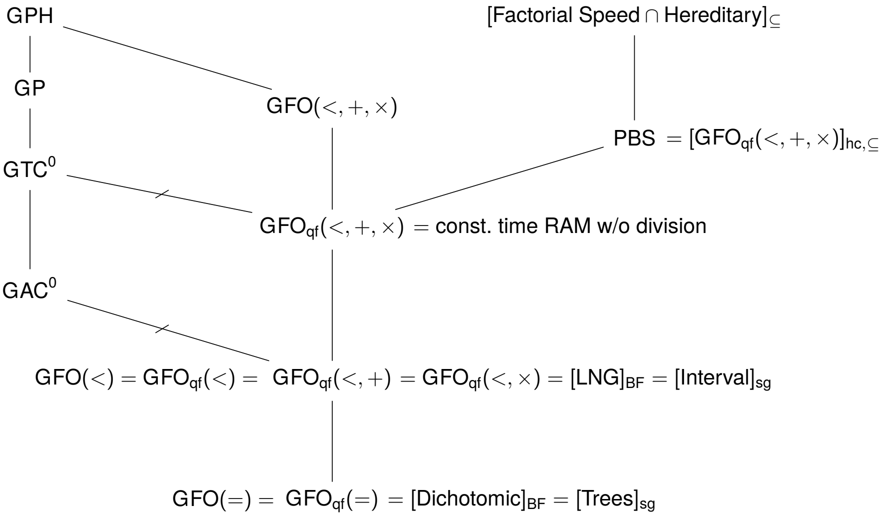
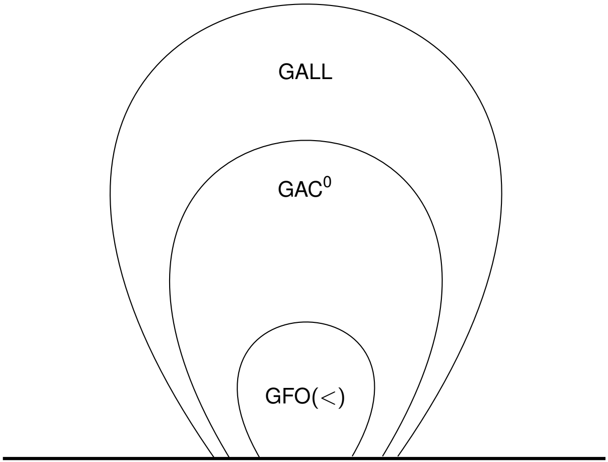
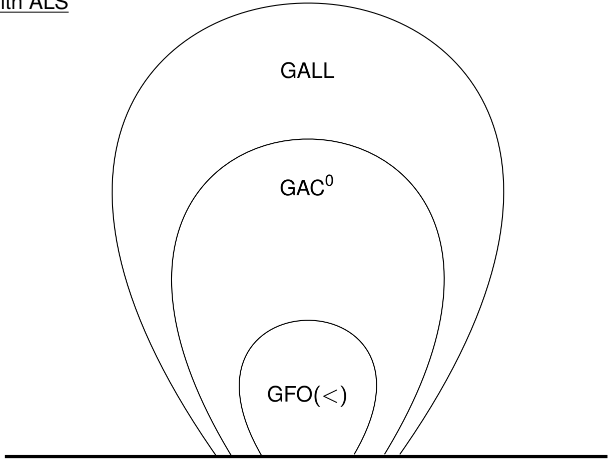


Refined Complexity of Graph Classes



Refined Complexity of Graph Classes

some classes with ALS



Refined Complexity of Graph Classes

some classes with ALS

planar graphs,
threshold graphs,
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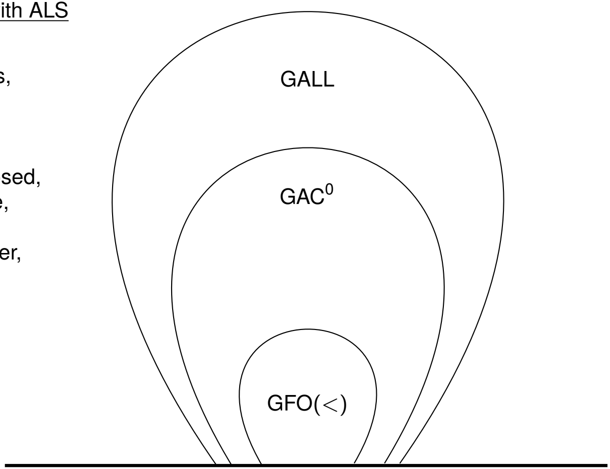
circle graphs,
proper minor-closed,
uniformly sparse,
b. boxicity,
b. interval number,
b. tree-width,
b. clique-width



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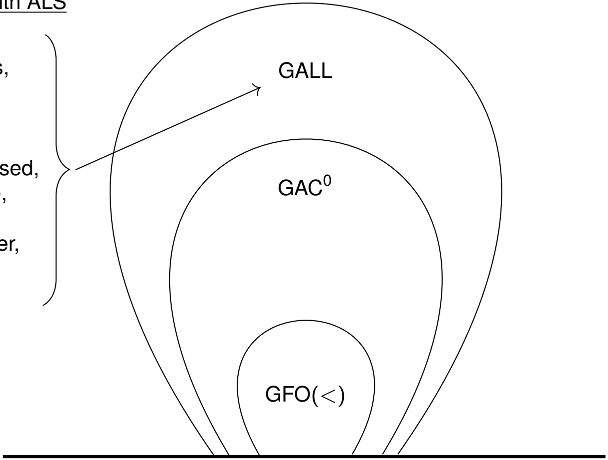

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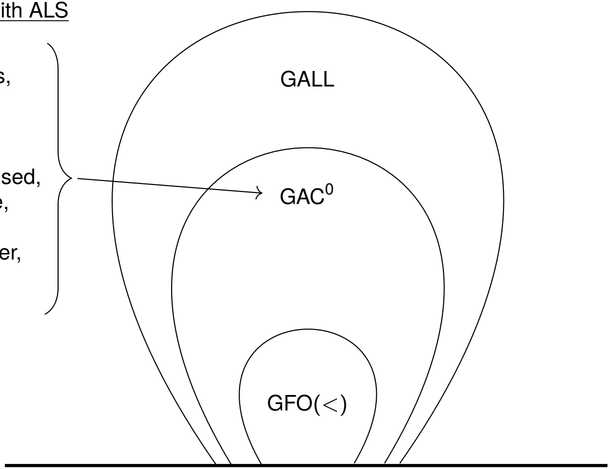

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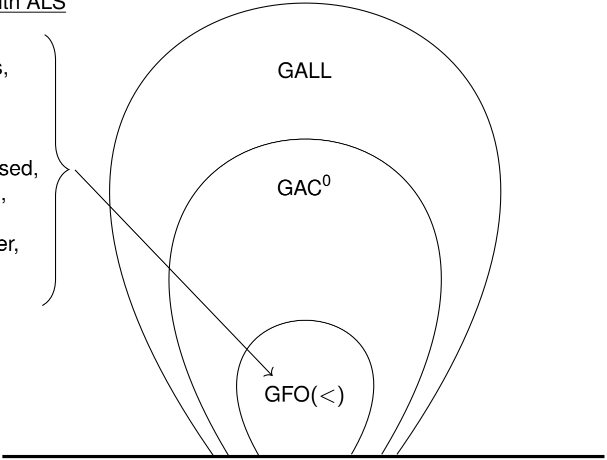

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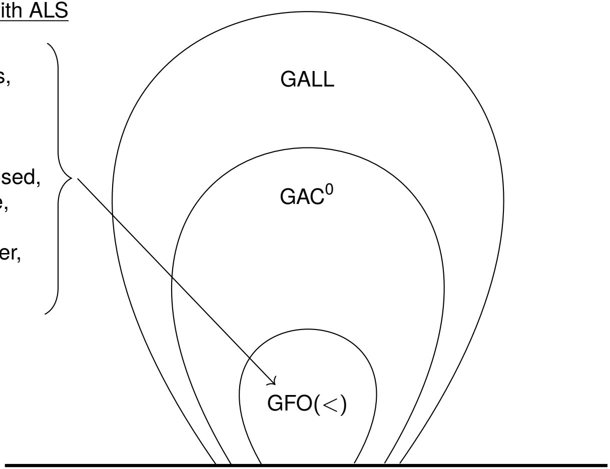

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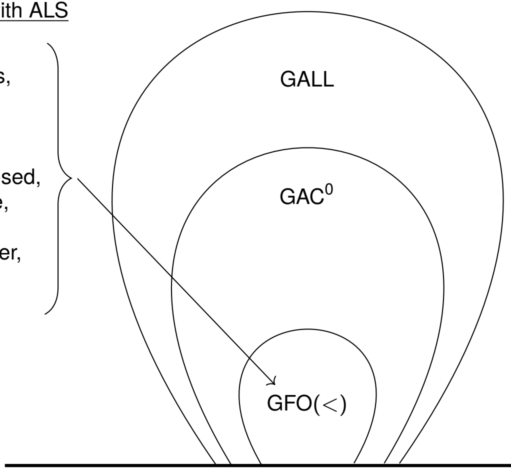

GFO($<$) contains everything* but clique-width

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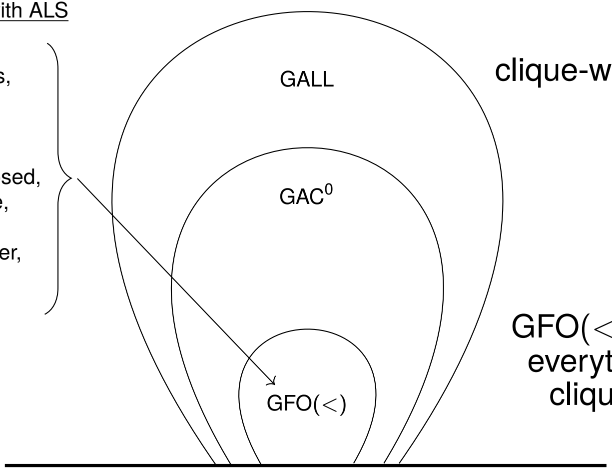

clique-width $\subseteq$ PBS?

GFO($<$) contains everything* but clique-width

Reduction for Graph Classes

- closure:

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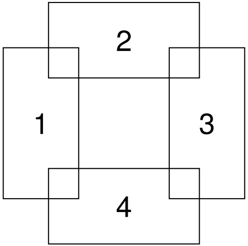

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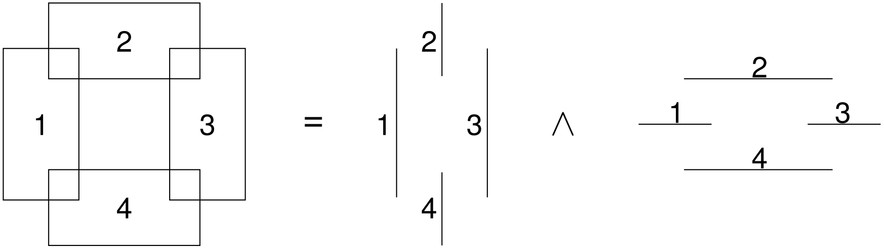

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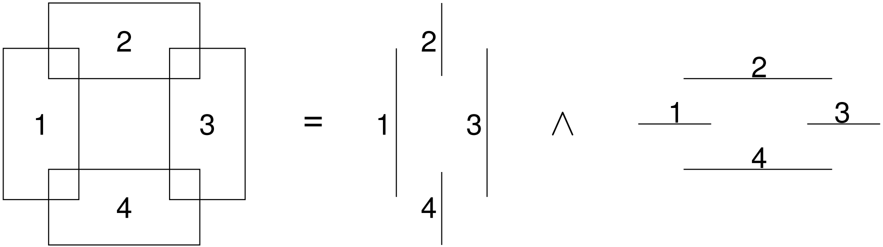
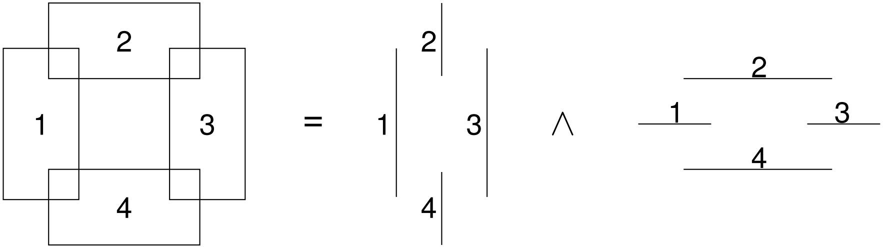
- box graphs $\leq_{\text{BF}}$ interval graphs



Algebraic Reduction $\leq_{BF}$

- boolean formulas for graph classes:



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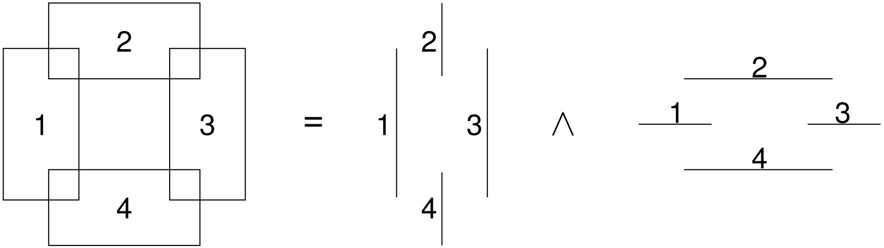

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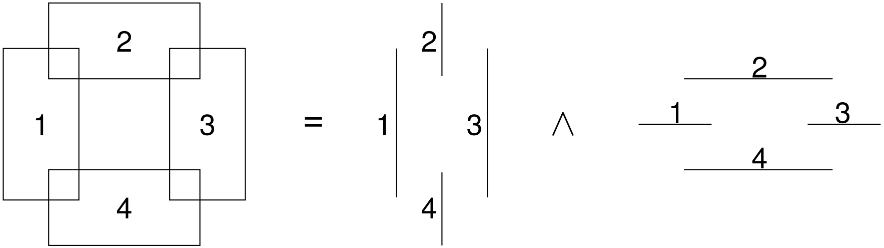

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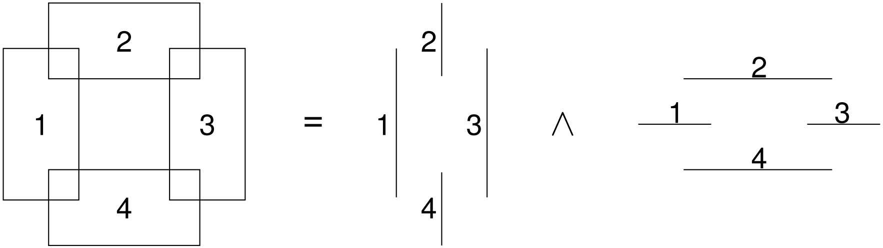
- $\mathcal{C} \wedge \mathcal{D} = \neg(\neg\mathcal{C} \vee \neg\mathcal{D})$



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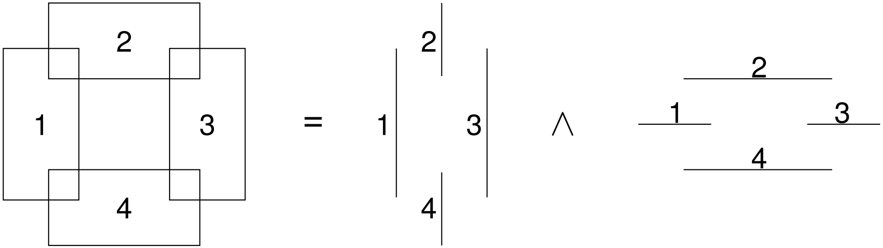

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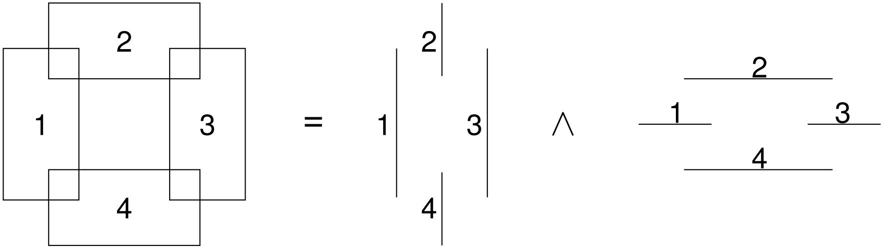

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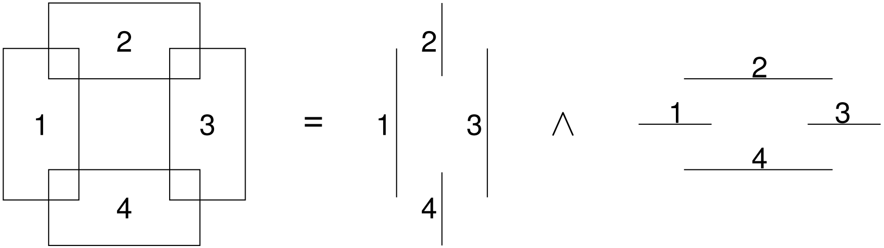

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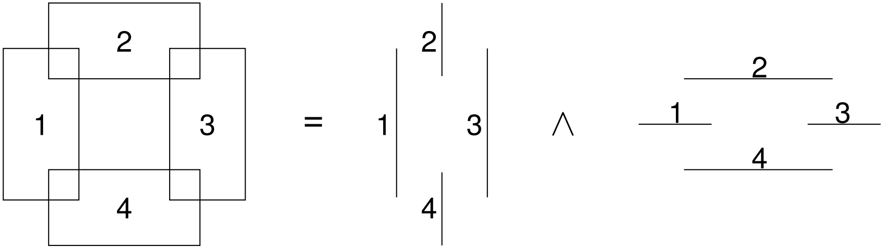

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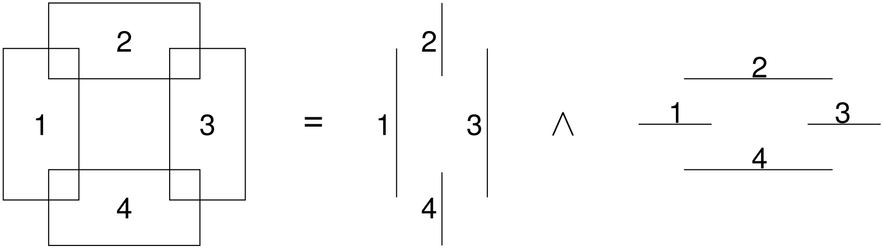

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